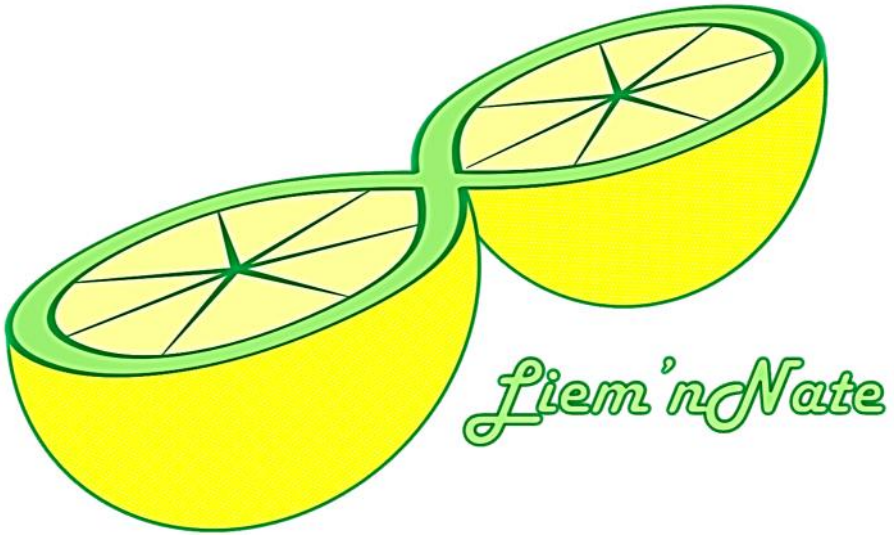
MfA Meets MTBoS

What's the Scenario?

Talking Scenario-Based Learning with



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Why We're Here

We are working on a "Pathway to Calculus" project with Math for America.

Designing a 3-year Common Core aligned curriculum with the AP Calculus Exam as one of the end goals.

Our scenario-based tasks have been implemented in multiple classrooms with positive feedback from teachers and students.

We would like to share how we create our tasks and the considerations involved. We hope you get something out of it!

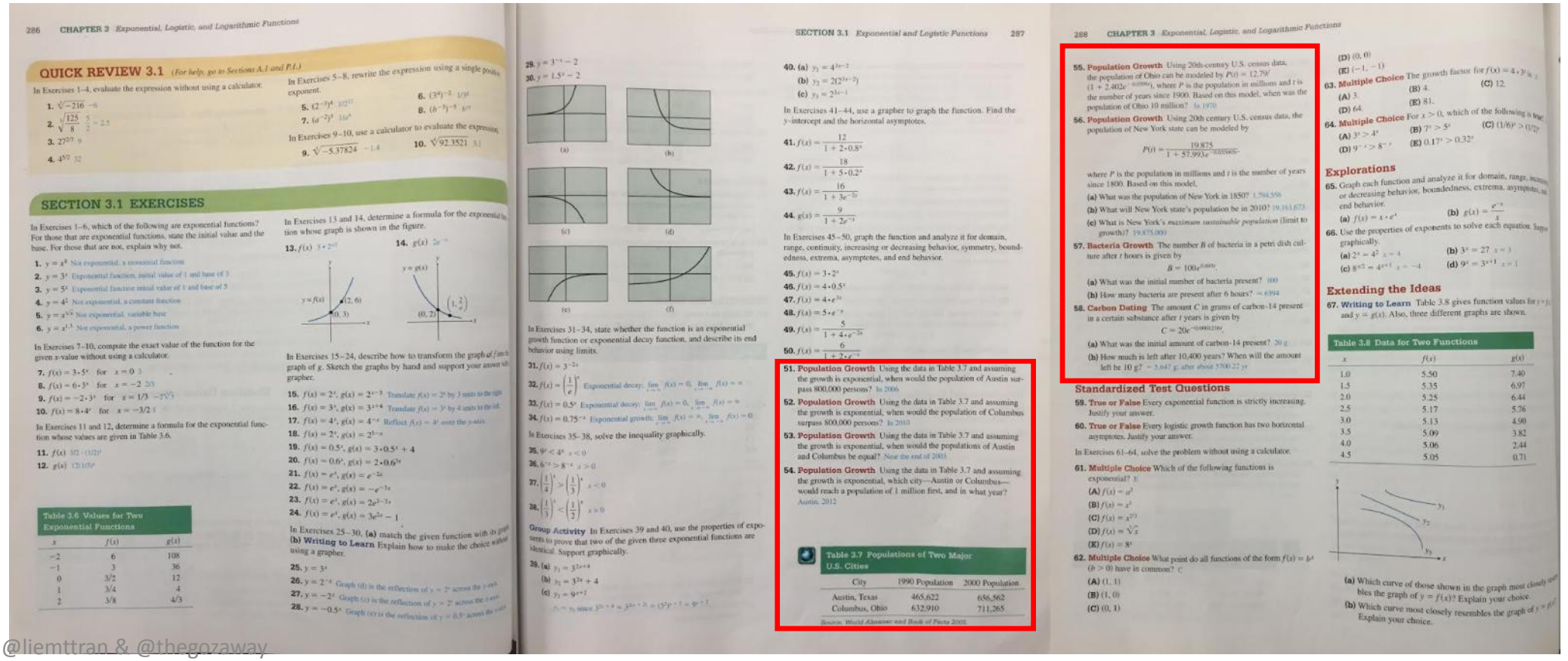
An Example from our Classroom:

Creating a Scenario-Based Exploration for Exponential Functions

@liemttran & @thegozaway

Pre-Calculus Unit 3: Exponential, Logistic and Logarithmic Functions

3.1: Exponential and Logistic



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QUICK REVIEW 3.1 *(For help, go to Sections A.1 and P.1.)*

In Exercises 1–4, evaluate the expression without using a calculator.

1. $\sqrt[3]{-216}$ −6
2. $\sqrt{\frac{125}{8}}$ $\frac{5}{2}$ = 2.5
3. $27^{2/3}$ 9
4. $4^{5/2}$ 32

In Exercises 5–8, rewrite the expression using a single positive exponent.

5. $(2^{-3})^4$ $1/2^{12}$
6. $(3^4)^{-2}$ $1/3^8$
7. $(a^{-2})^3$ $1/a^6$
8. $(b^{-3})^{-5}$ b^{15}

In Exercises 9–10, use a calculator to evaluate the expression.

9. $\sqrt[3]{-5.37824}$ −1.4
10. $\sqrt[5]{92.3521}$ 3.1

SECTION 3.1 EXERCISES

In Exercises 1–6, which of the following are exponential functions? For those that are exponential functions, state the initial value and the base. For those that are not, explain why not.

1. $y = x^8$ Not exponential, a monomial function
2. $y = 3^x$ Exponential function, initial value of 1 and base of 3
3. $y = 5^x$ Exponential function initial value of 1 and base of 5
4. $y = 4^2$ Not exponential, a constant function
5. $y = x^{\sqrt{2}}$ Not exponential, variable base
6. $y = x^{1.3}$ Not exponential, a power function

In Exercises 7–10, compute the exact value of the function for the given x-value without using a calculator.

7. $f(x) = 3 \cdot 5^x$ for $x = 0$ 3
8. $f(x) = 6 \cdot 3^x$ for $x = -2$ 2/3
9. $f(x) = -2 \cdot 3^x$ for $x = 1/3$ $-2\sqrt[3]{3}$
10. $f(x) = 8 \cdot 4^x$ for $x = -3/2$ 1

In Exercises 11 and 12, determine a formula for the exponential function whose values are given in Table 3.6.

11. $f(x)$ $3/2 \cdot (1/2)^x$
12. $g(x)$ $12(3/5)^x$

Table 3.6 Values for Two Exponential Functions

x	$f(x)$	$g(x)$
−2	6	108
−1	3	36
0	3/2	12
1	3/4	4
2	3/8	4/3

In Exercises 13 and 14, determine a formula for the exponential function whose graph is shown in the figure.

13. $f(x)$ $3 \cdot 2^{x/2}$

14. $g(x)$ $2e^{-x}$

In Exercises 15–24, describe how to transform the graph of f into the graph of g. Sketch the graphs by hand and support your answer with a grapher.

15. $f(x) = 2^x$, $g(x) = 2^{x-3}$ Translate $f(x) = 2^x$ by 3 units to the right
16. $f(x) = 3^x$, $g(x) = 3^{x+4}$ Translate $f(x) = 3^x$ by 4 units to the left
17. $f(x) = 4^x$, $g(x) = 4^{-x}$ Reflect $f(x) = 4^x$ over the y-axis
18. $f(x) = 2^x$, $g(x) = 2^{5-x}$
19. $f(x) = 0.5^x$, $g(x) = 3 \cdot 0.5^x + 4$
20. $f(x) = 0.6^x$, $g(x) = 2 \cdot 0.6^{3x}$
21. $f(x) = e^x$, $g(x) = e^{-2x}$
22. $f(x) = e^x$, $g(x) = -e^{-3x}$
23. $f(x) = e^x$, $g(x) = 2e^{3-3x}$
24. $f(x) = e^x$, $g(x) = 3e^{2x} - 1$

In Exercises 25–30, **(a)** match the given function with its graph and **(b) Writing to Learn** Explain how to make the choice without using a grapher.

25. $y = 3^x$
26. $y = 2^{-x}$ Graph (d) is the reflection of $y = 2^x$ across the y-axis
27. $y = -2^x$ Graph (c) is the reflection of $y = 2^x$ across the x-axis
28. $y = -0.5^x$ Graph (e) is the reflection of $y = 0.5^x$ across the x-axis

SECTION 3.1 Exponential and Logistic Functions 287

29. $y = 3^{-x} - 2$
30. $y = 1.5^x - 2$

In Exercises 31–34, state whether the function is an exponential growth function or exponential decay function, and describe its end behavior using limits.

31. $f(x) = 3^{-2x}$
32. $f(x) = \left(\frac{1}{e}\right)^x$ Exponential decay: $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = \infty$
33. $f(x) = 0.5^x$ Exponential decay: $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = \infty$
34. $f(x) = 0.75^{-x}$ Exponential growth: $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 0$

In Exercises 35–38, solve the inequality graphically.

35. $9^x < 4^x$ $x < 0$
36. $6^{-x} > 8^{-x}$ $x > 0$
37. $\left(\frac{1}{4}\right)^x > \left(\frac{1}{3}\right)^x$ $x < 0$
38. $\left(\frac{1}{3}\right)^x < \left(\frac{1}{2}\right)^x$ $x > 0$

Group Activity In Exercises 39 and 40, use the properties of exponents to prove that two of the given three exponential functions are identical. Support graphically.

39. (a) $y_1 = 3^{2x+4}$
(b) $y_2 = 3^{2x} + 4$
(c) $y_3 = 9^{x+2}$
$y_1 = y_3$ since $3^{2x+4} = 3^{2x+2} = (3^2)^{x+2} = 9^{x+2}$

40. (a) $y_1 = 4^{3x-2}$
(b) $y_2 = 2(2^{3x-2})$
(c) $y_3 = 2^{3x-1}$

In Exercises 41–44, use a grapher to graph the function. Find the y-intercept and the horizontal asymptotes.

41. $f(x) = \frac{12}{1 + 2 \cdot 0.8^x}$
42. $f(x) = \frac{18}{1 + 5 \cdot 0.2^x}$
43. $f(x) = \frac{16}{1 + 3e^{-2x}}$
44. $g(x) = \frac{9}{1 + 2e^{-x}}$

In Exercises 45–50, graph the function and analyze it for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

45. $f(x) = 3 \cdot 2^x$
46. $f(x) = 4 \cdot 0.5^x$
47. $f(x) = 4 \cdot e^{2x}$
48. $f(x) = 5 \cdot e^{-x}$
49. $f(x) = \frac{5}{1 + 4 \cdot e^{-2x}}$
50. $f(x) = \frac{6}{1 + 2 \cdot e^{-x}}$

51. **Population Growth** Using the data in Table 3.7 and assuming the growth is exponential, when would the population of Austin surpass 800,000 persons? In 2006
52. **Population Growth** Using the data in Table 3.7 and assuming the growth is exponential, when would the population of Columbus surpass 800,000 persons? In 2010
53. **Population Growth** Using the data in Table 3.7 and assuming the growth is exponential, when would the populations of Austin and Columbus be equal? Near the end of 2003
54. **Population Growth** Using the data in Table 3.7 and assuming the growth is exponential, which city—Austin or Columbus—would reach a population of 1 million first, and in what year? Austin, 2012

Table 3.7 Populations of Two Major U.S. Cities

City	1990 Population	2000 Population
Austin, Texas	465,622	656,562
Columbus, Ohio	632,910	711,265

Source: World Almanac and Book of Facts 2003.

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55. **Population Growth** Using 20th-century U.S. census data, the population of Ohio can be modeled by $P(t) = 12.79/(1 + 2.402e^{-0.0390t})$, where P is the population in millions and t is the number of years since 1900. Based on this model, when was the population of Ohio 10 million? In 1970
56. **Population Growth** Using 20th century U.S. census data, the population of New York state can be modeled by

$$P(t) = \frac{19.875}{1 + 57.993e^{-0.0359t}},$$

where P is the population in millions and t is the number of years since 1800. Based on this model,
(a) What was the population of New York in 1850? 1,584,558
(b) What will New York state's population be in 2010? 19,161,673
(c) What is New York's *maximum sustainable population* (limit to growth)? 19,875,000
57. **Bacteria Growth** The number B of bacteria in a petri dish culture after t hours is given by

$$B = 100e^{0.69t}.$$

(a) What was the initial number of bacteria present? 100
(b) How many bacteria are present after 6 hours? ≈ 6394
58. **Carbon Dating** The amount C in grams of carbon-14 present in a certain substance after t years is given by

$$C = 20e^{-0.0001210t}.$$

(a) What was the initial amount of carbon-14 present? 20 g
(b) How much is left after 10,400 years? When will the amount left be 10 g? ≈ 3.647 g; after about 5700.22 yr

Standardized Test Questions

59. **True or False** Every exponential function is strictly increasing. Justify your answer.
60. **True or False** Every logistic growth function has two horizontal asymptotes. Justify your answer.

In Exercises 61–64, solve the problem without using a calculator.

61. **Multiple Choice** Which of the following functions is exponential? E
(A) $f(x) = a^3$
(B) $f(x) = x^5$
(C) $f(x) = x^{2/3}$
(D) $f(x) = \sqrt{x}$
(E) $f(x) = 8^x$
62. **Multiple Choice** What point do all functions of the form $f(x) = b^x$ ($b > 0$) have in common? C
(A) (1, 1)
(B) (1, 0)
(C) (0, 1)
(D) (0, 0)
(E) (−1, −1)
63. **Multiple Choice** The growth factor for $f(x) = 4 \cdot 3^x$ is
(A) 3.
(B) 4.
(C) 12.
(D) 64.
(E) 81.
64. **Multiple Choice** For $x > 0$, which of the following is true?
(A) $3^x > 4^x$
(B) $7^x > 5^x$
(C) $(1/6)^x > (1/7)^x$
(D) $9^{-x} > 8^{-x}$
(E) $0.17^x > 0.32^x$

Explorations

65. Graph each function and analyze it for domain, range, increasing or decreasing behavior, boundedness, extrema, asymptotes, and end behavior.
(a) $f(x) = x \cdot e^x$
(b) $g(x) = \frac{e^{-x}}{4}$
66. Use the properties of exponents to solve each equation. Support graphically.
(a) $2^x = 4^2$ $x = 4$
(b) $3^x = 27$ $x = 3$
(c) $8^{x/2} = 4^{x+1}$ $x = -4$
(d) $9^x = 3^{x+1}$ $x = 1$

Extending the Ideas

67. **Writing to Learn** Table 3.8 gives function values for $y = f(x)$ and $y = g(x)$. Also, three different graphs are shown.

Table 3.8 Data for Two Functions

x	$f(x)$	$g(x)$
1.0	5.50	7.40
1.5	5.35	6.97
2.0	5.25	6.44
2.5	5.17	5.76
3.0	5.13	4.90
3.5	5.09	3.82
4.0	5.06	2.44
4.5	5.05	0.71

(a) Which curve of those shown in the graph most closely resembles the graph of $y = f(x)$? Explain your choice.
(b) Which curve most closely resembles the graph of $y = g(x)$? Explain your choice.

57. Bacteria Growth The number B of bacteria in a petri dish culture after t hours is given by

$$B = 100e^{0.693t}.$$

(a) What was the initial number of bacteria present? 100

(b) How many bacteria are present after 6 hours? ≈ 6394

What do we like about this problem?

What don't we like?

From Word Problems to Scenario-Based Tasks

The “classic word problem” presents a scenario that has potential for mathematical modeling, reasoning and thinking and *then promptly ignores that scenario in favor of recognition and math doing*.

Our goal is to create a more accessible scenario and build a rich mathematical exploration around it.

When successful this results in a one “word problem” per class pace.

The number of bacteria in a petri dish is growing in such a way that its population quadruples every hour. If there are initially 20 bacteria in the population, *how much time will pass before there are 10,000 bacteria in the dish?*

Do we (the teachers) want to talk about bacteria for the next two hours?
Will our students want to talk about bacteria?
Can we make estimations?
What is reasonable? How will we know if our answers make sense?
Can we make jokes about the material? Can our students?
Are there interesting questions with answers we care about?

NO!

At the start of 2016 the up and coming Math Education duo **@LiemnNate** had exactly 20 followers on Twitter. Experts predict this account is about to blow up as Liem and Nate become well known in classrooms and educational circles. In fact, it is estimated that the number of followers will quadruple every year for the next few years.

How does this become a 2 hour lesson?

What would you ask?

Where could we go with this?

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How does this become a 2 hour lesson?

What we asked…

The Basics

How many followers will @LiemnNate have at the start of 2017?

Write an equation for *L(t)*, the predicted number of followers @LiemnNate will have *t* years from 1/1/16.

Evaluate *L(2)*. What does this answer represent?

At the beginning of which year will @LiemnNate to reach exactly 1,280 followers?

Rational Exponents: Patterns?

Evaluate $L\left(\frac{3}{2}\right)$. What does this answer represent? Does this answer make sense? Explain.

How many followers is @LiemnNate predicted to have in two and a half years?

Evaluate *L(2.25)*. What does your answer represent?

How many twitter followers is @LiemnNate predicted to have at the end of this month? Show the work that leads to your answer. Explain why your answer seems reasonable (or does it?).

The Essential Question

(Yes, we care!)

Right now the math education guru @ddmeyer has 46.3K followers. How long will it take @LiemnNate to obtain that many followers?

&

The Super Follow Up

(This could take a while!)

In January 2014 Dan had 23.3K followers and in January 2015 he was up to 33.6K followers. How long will it take for @LiemnNate to actually catch up to Dan Meyer? Justify your answer by providing the necessary mathematical evidence.

Always Fun…

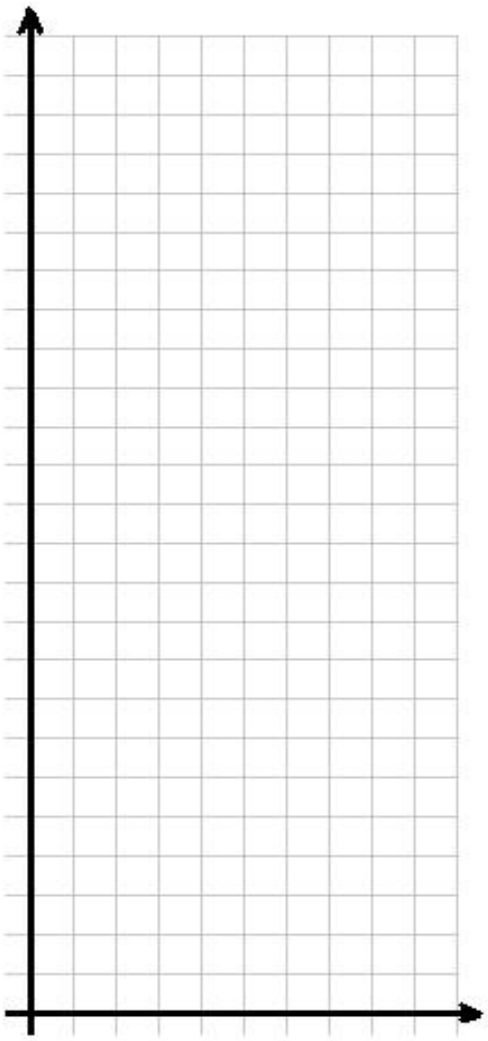
What is a reasonable domain for *L*? How did you come to this conclusion?

Is *L* discrete or continuous? Explain your choice.

Which leads to…

The Graph!

Sketch a graph of $y=L(t)$.



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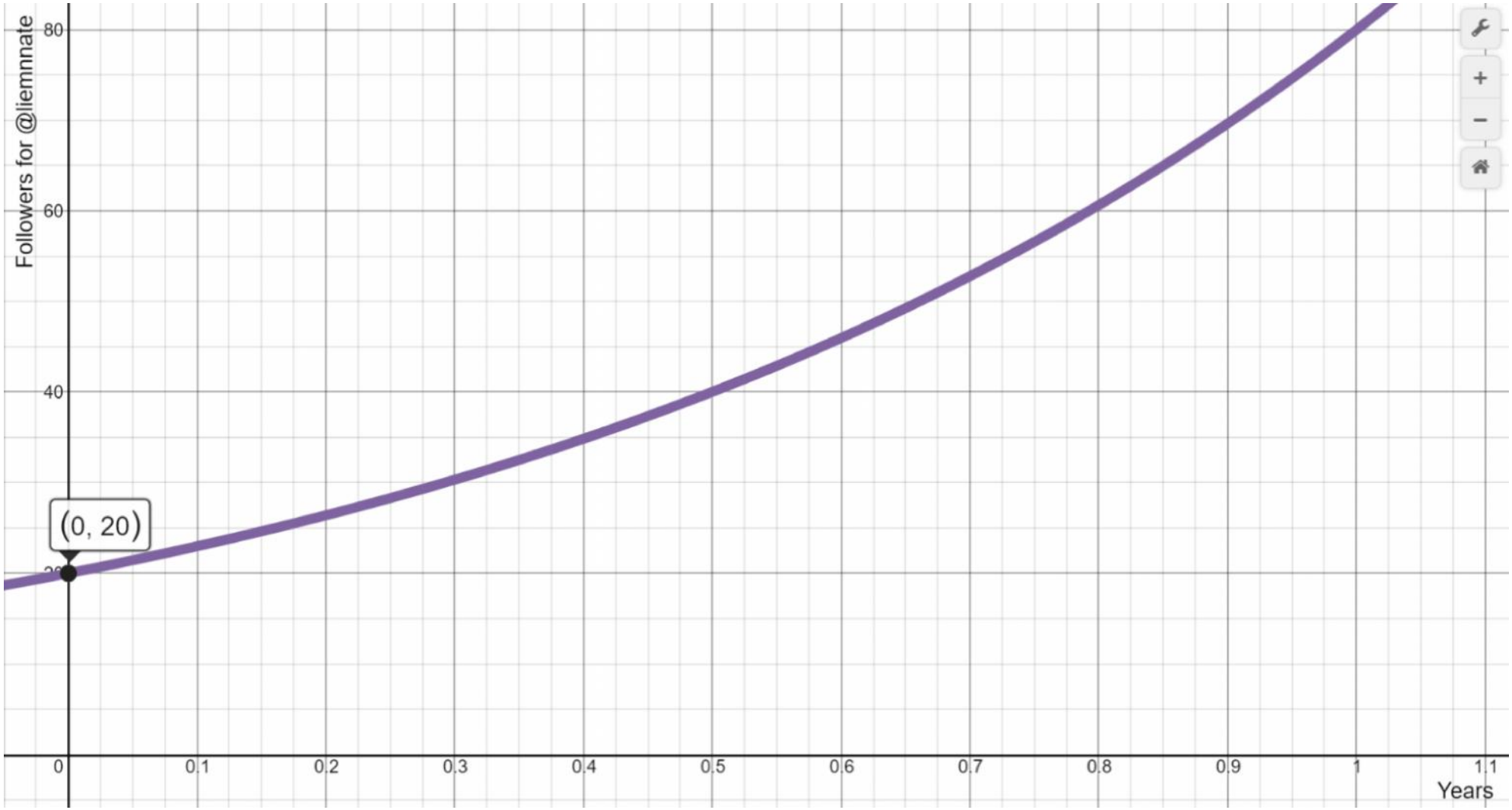

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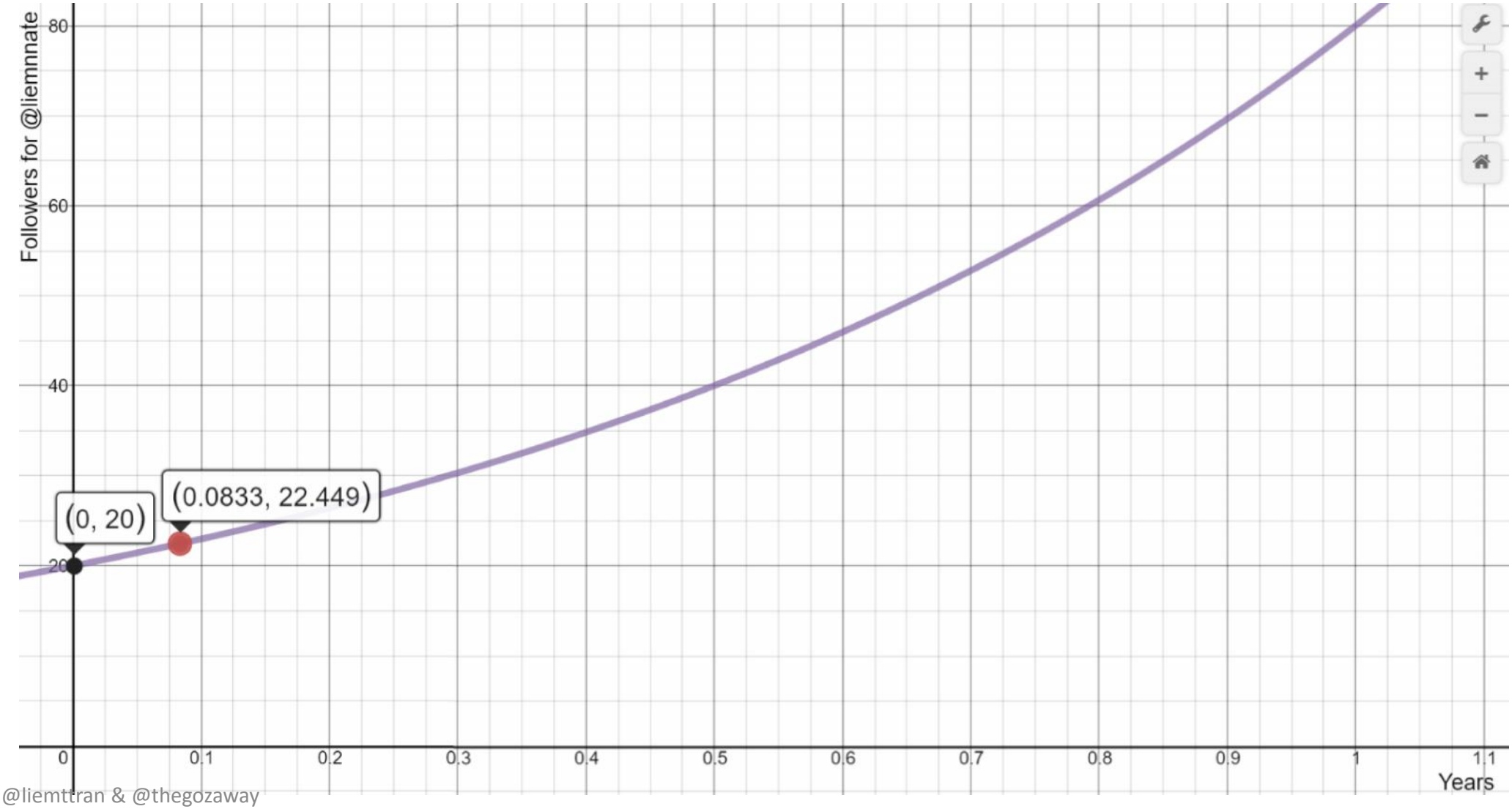

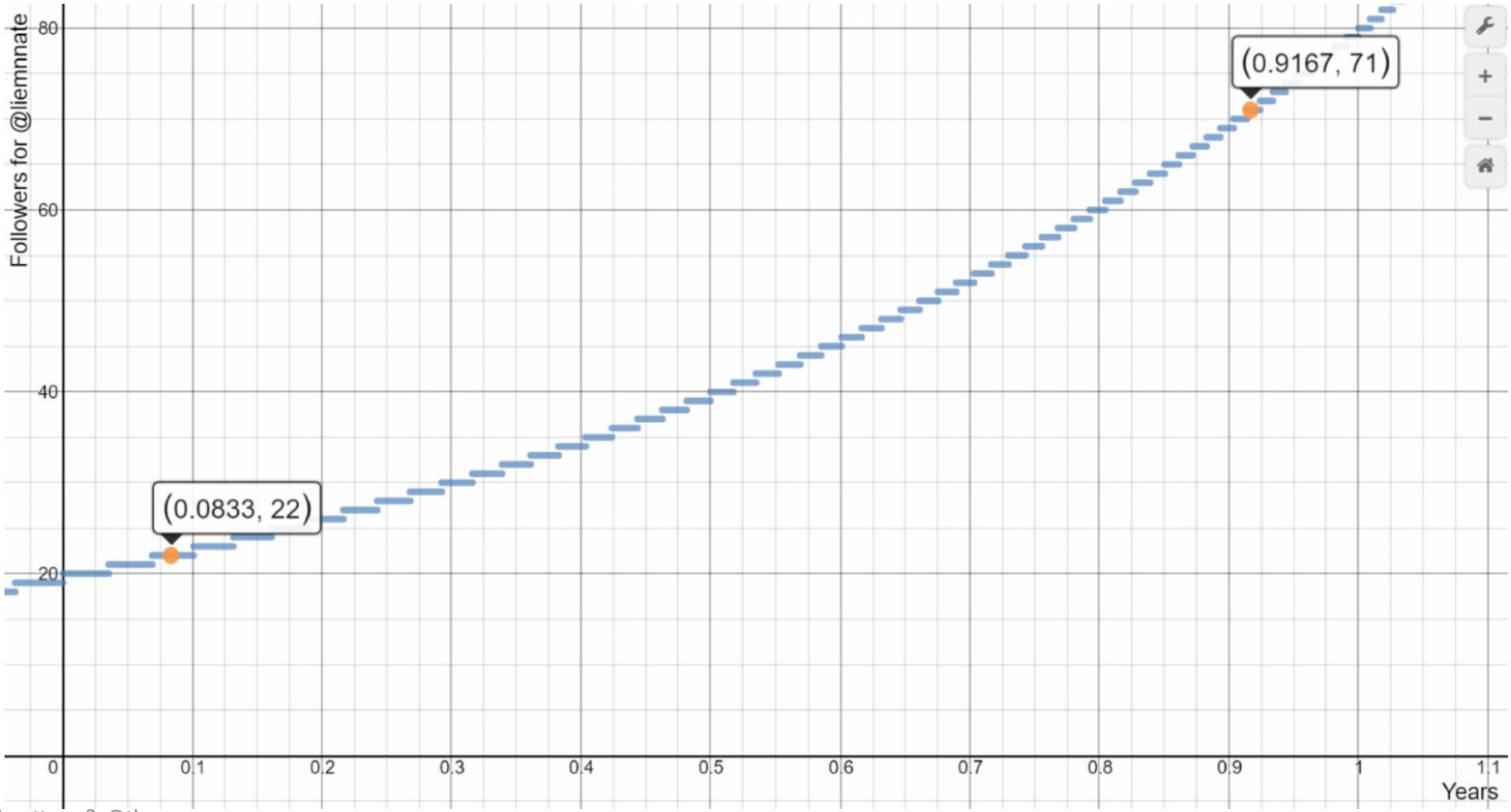

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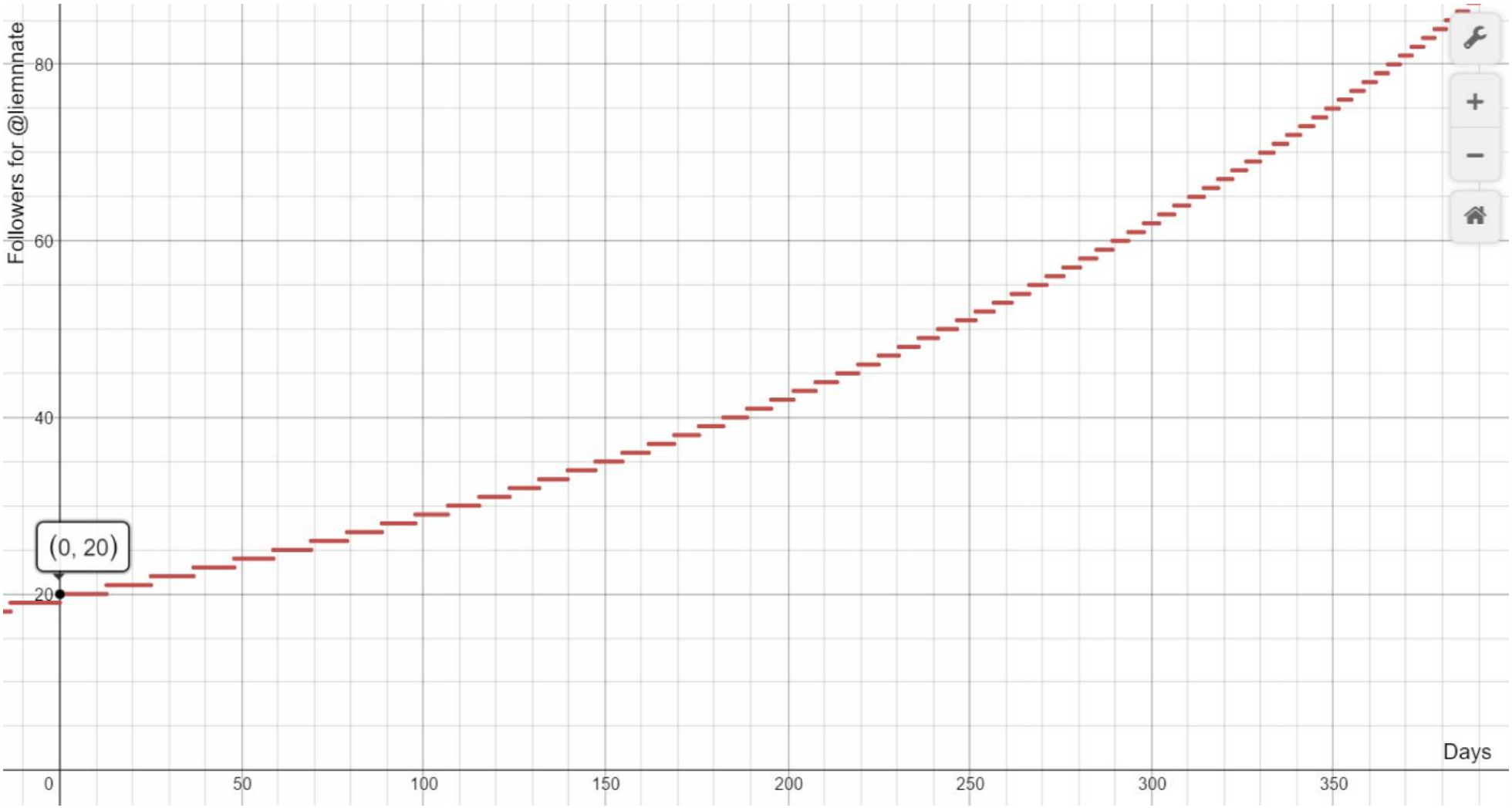

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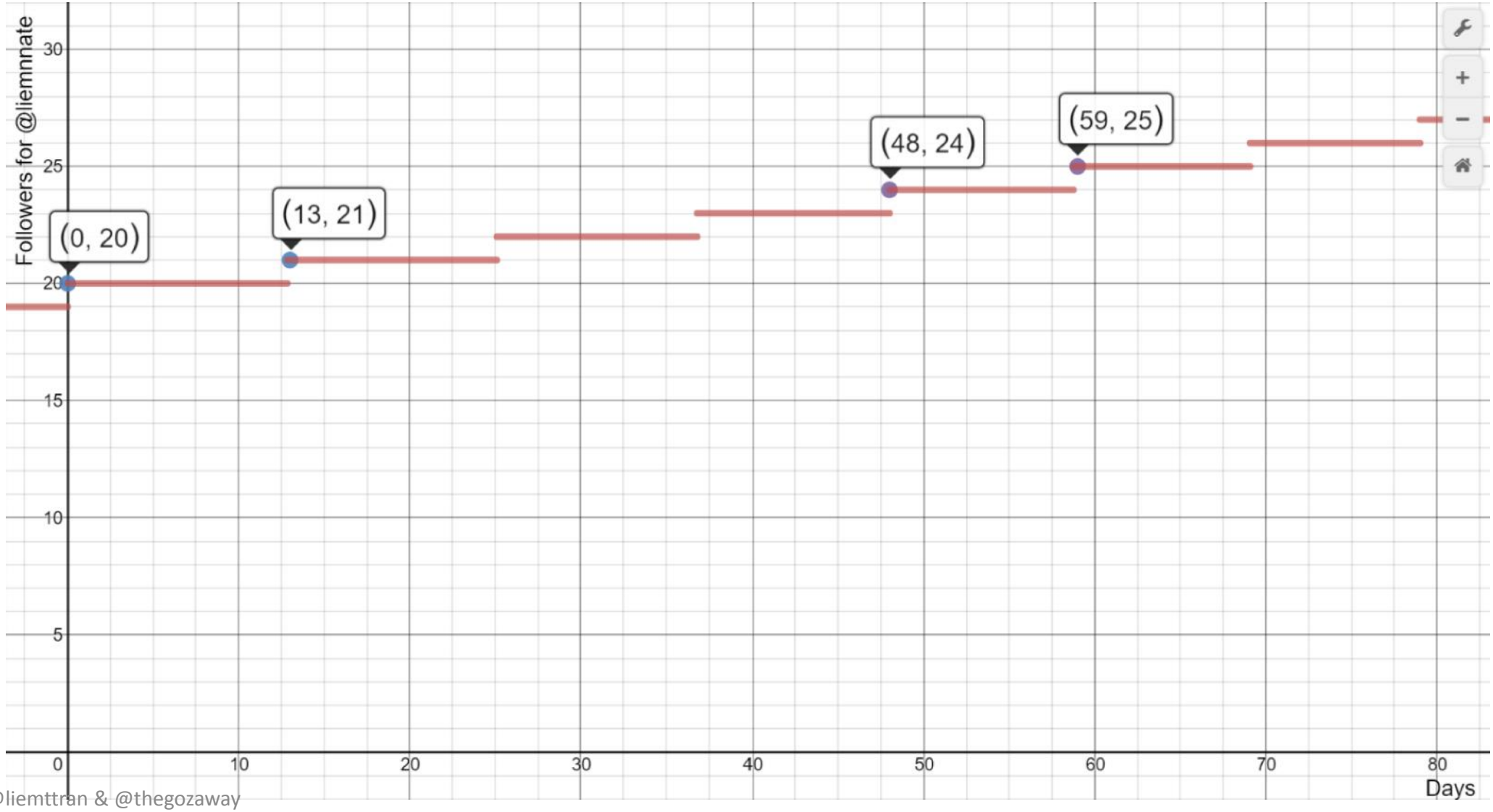

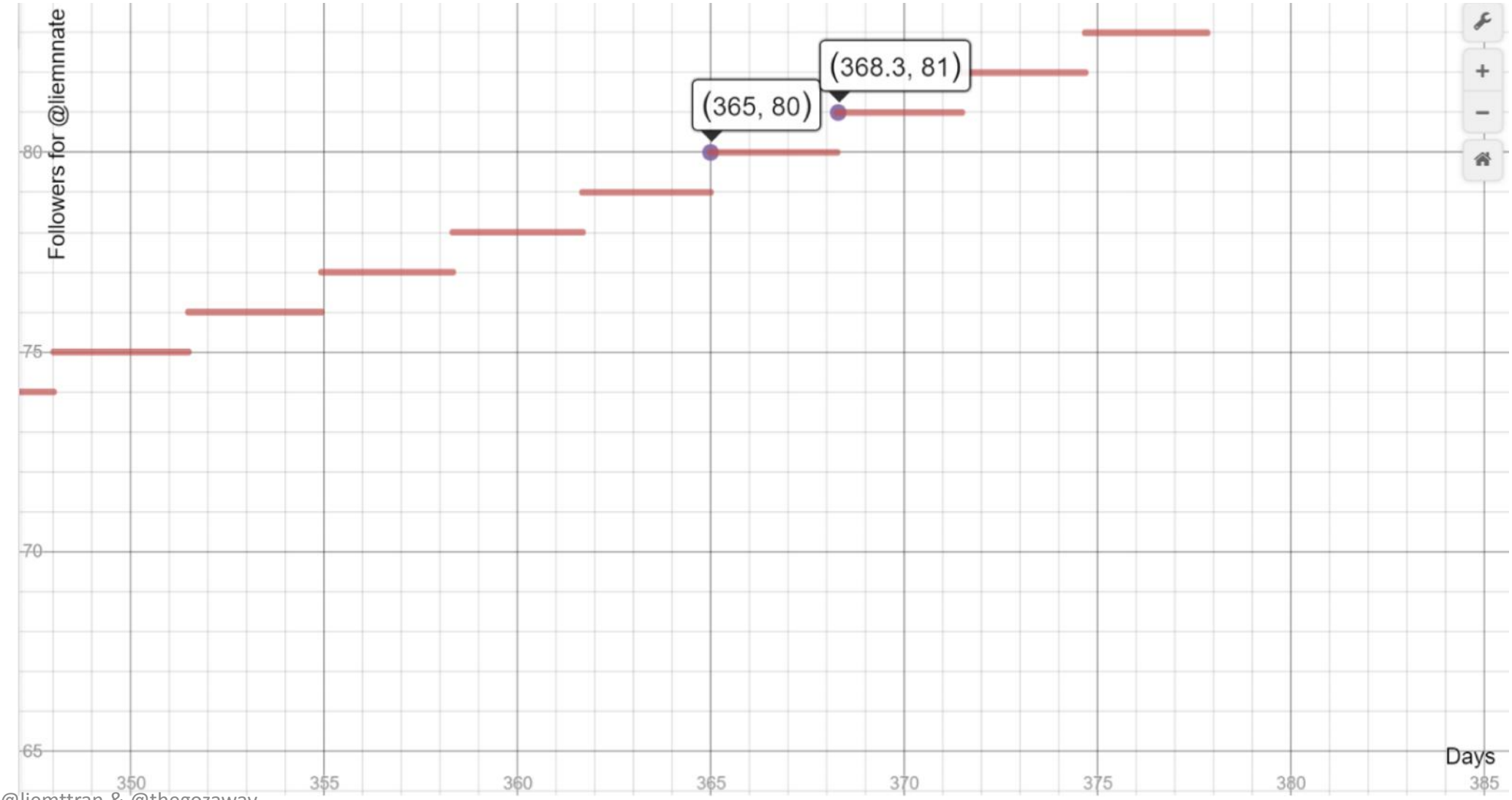

@liemttran & @thegozaway

Negative Exponents: Going Back in Time!!

Whether or not you included negative values in your domain, use your equation to evaluate *L(-1).* What does this answer represent? Does it make sense?

One More for the Road: Disrupting the Model

What if everyone in this class goes home today and follows @LiemnNate on twitter? *(You know you want to!)*

Will this ruin *L* as a model for predicting followers? Should we change the model? Explain why or why not.

If so, how should we change it?

Try this in Your Classroom!

- **Find a classic "word problem" for your topic.**
- **Isolate the scenario and take out the questions.**
- **Make the content more relevant/accessible/enjoyable/hilarious for *you and your students*.**
- **Open the problem up to exploration.**
- **Think of all the questions (mostly but not all mathy) related to the topic and ask as many as you can. Keep an essential goal in mind.**
- **Spiral, unload, and embed. The skills will surface on their own.**

Contact Information

Session Handouts: bit.do/whatsthescenario
Session Evaluation: bit.do/liemnnate

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Oh and @LiemnNate is real. But we are laying low for now.